UNITÉ DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France

Tél.:(1) 39 63 55 11

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ON THE APPROXIMATION OF GENERAL SHELL PROBLEMS BY THE CLOUGH-JOHNSON FLAT PLATE ELEMENTS PART 3 : THE IMPLEMENTATION

Michel BERNADOU
Yves DUCATEL

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ON THE APPROXIMATION OF GENERAL SHELL PROBLEMS BY THE CLOUGH-JOHNSON FLAT PLATE ELEMENTS

PART 3 : THE IMPLEMENTATION

Michel BERNADOU[1] - Yves DUCATEL[2]

Summary : In this series of reports, we give an account of some results obtained in the approximation of *general shell problems* by the CLOUGH-JOHNSON *flat plate elements*. The first part is concerned by the study of the *compatibility equations*. In the second part, we deliver several interesting results valid for general shells and we prove the "pseudo-convergence" of the method for a class of shallow shells. Then, this careful study allows us to introduce a perturbation of this approximation and thus, to propose a new method which is convergent for general shells. Finally, in the third part, we describe in details how to implement the CLOUGH-JOHNSON method.

SUR L'APPROXIMATION DE PROBLEMES GENERAUX DE COQUES PAR DES METHODES D'ELEMENTS FINIS PLATS DE CLOUGH ET JOHNSON

PARTIE 3 : IMPLEMENTATION

Résumé : Dans cette série de rapports, nous rassemblons les divers résultats obtenus dans l'approximation de *problèmes généraux de coques* par les éléments finis plats (de plaques) de CLOUGH et JOHNSON. La première partie est relative à l'étude des équations de *compatibilité*. Dans la seconde partie, nous donnons plusieurs résultats intéressants valables pour des coques générales, puis nous démontrons la pseudo-convergence de la méthode pour une classe de coques peu profondes. Cette étude détaillée nous permet alors d'introduire une perturbation de cette approximation et ainsi, de proposer une nouvelle méthode qui converge pour des coques générales. Finalement, dans la troisième partie, nous détaillons la marche à suivre pour implémenter la méthode de CLOUGH et JOHNSON.

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(1) INRIA, Domaine de Voluceau, Rocquencourt, B.P. 105, 78153 Le Chesnay

(2) Université de Poitiers, Département de Mathématiques,
40, avenue du Recteur Pineau, 86022 Poitiers Cedex

PAPIER RÉCUPÉRÉ ET RECYCLÉ

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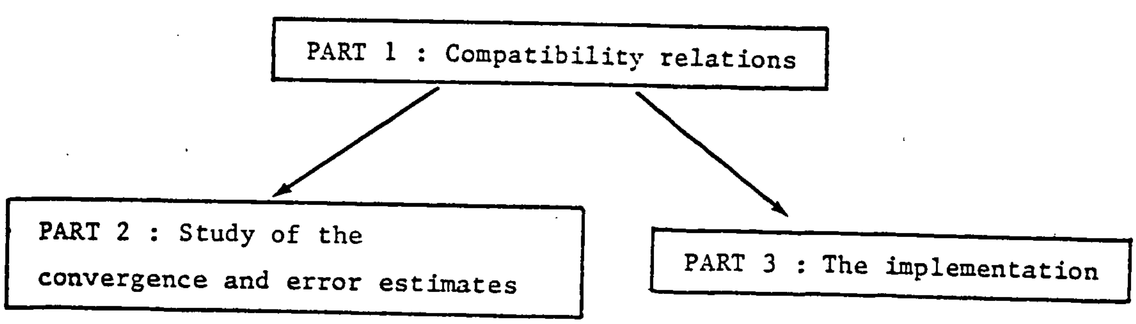
HOW TO READ THE PAPER

All the paper is divided in three parts. In the first one the reader can find all the basis. Then, *part 2 and part 3 can be read independtly* depending of the interest of everyone. In other words we can summarize these possibilities in the following "flow shart" :



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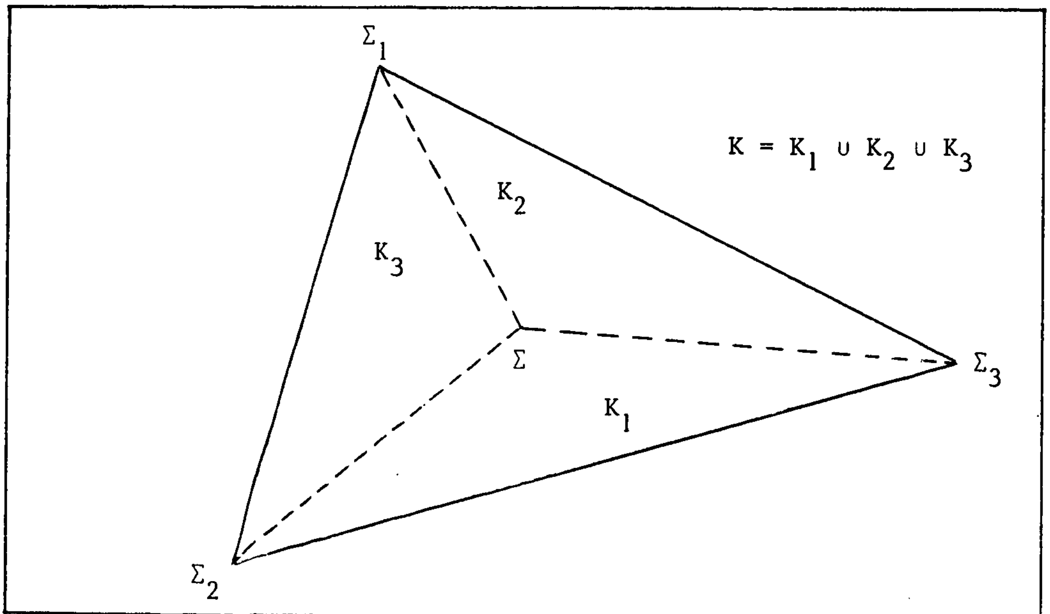
7 - IMPLEMENTATION OF THE REDUCED HSIEH-CLOUGH-TOCHER TRIANGLE

Orientation : In this paragraph we record the main features of the implementation of the triangle of type 1 and of the reduced HCT triangle. For more details we refer to BERNADOU-BOISSERIE [2] and BERNADOU-HASSAN [20].

7.1 - Definition of the reduced HCT triangle

Let K be a triangle with vertices Σ_1, Σ_2, Σ_3 and barycenter Σ. Let K_i be the triangles with vertices Σ, Σ_{i+1}, Σ_{i-1} , $i=1,2,3$. Here and subsequently the indices belong to $\{1,2,3\}$ modulo 3.

Then, the definition of the reduced HCT triangle (K,P_K,Σ_K) is condensed in Fig. 7.1.1. Let us record that P_K and Σ_K denote the discrete space and the set of degrees of freedom, respectively.



$$P_K = \{p \in \mathscr{C}^1(K) \ ; \ r_i = p|_{K_i} \in P_3(K_i) \quad , \quad 1 \le i \le 3 \ ,$$

$$\partial_\nu p \in P_1(K') \text{ pour chaque côté } K' \text{ de } K\}$$

$$\dim P_K = 9$$

$$\Sigma_K = \{p(\Sigma_i), Dp(\Sigma_i)(\Sigma_{i-1}-\Sigma_i), Dp(\Sigma_i)(\Sigma_{i+1}-\Sigma_i) \quad , \quad 1 \le i \le 3\}$$

Figure 7.1.1 : The reduced HCT-triangle

7.2 - Barycentric coordinates - Cartesian coordinates

Let there be given an orthonormal fixed reference system of the plane $\mathbb{R}^2$, $(0,\vec{e}_1,\vec{e}_2)$. The coordinates of the point P are denoted (ξ^1,ξ^2), i.e.,

$$\vec{OP} = \xi^1\vec{e}_1 + \xi^2\vec{e}_2 .$$

Let K be a triangle with vertices $\Sigma_1(\xi_1^1,\xi_1^2)$, $\Sigma_2(\xi_2^1,\xi_2^2)$, $\Sigma_3(\xi_3^1,\xi_3^2)$. We assume $\overrightarrow{\Sigma_1\Sigma_2} \times \overrightarrow{\Sigma_1\Sigma_3} \neq \vec{0}$.

Then, the cartesian coordinates (ξ^1,ξ^2) of the point P are linked to the barycentric coordinates $(\lambda_1,\lambda_2,\lambda_3)$ by the relations

$$\left.\begin{aligned} \xi^1 &= \lambda_i\xi_1^1 + \lambda_2\xi_2^1 + \lambda_3\xi_3^1 , \\ \xi^2 &= \lambda_1\xi_1^2 + \lambda_2\xi_2^2 + \lambda_3\xi_3^2 . \end{aligned}\right\} \tag{7.2.1}$$

Let us record that

$$1 = \lambda_1 + \lambda_2 + \lambda_3 . \tag{7.2.2}$$

Conversely,

$$\left.\begin{aligned} \lambda_1 &= \frac{(\xi^1-\xi_2^1)(\xi_2^2-\xi_3^2) + (\xi_3^1-\xi_2^1)(\xi^2-\xi_2^2)}{\Delta} \\ \lambda_2 &= \frac{(\xi^1-\xi_3^1)(\xi_3^2-\xi_1^2) + (\xi_1^1-\xi_3^1)(\xi^2-\xi_3^2)}{\Delta} \\ \lambda_3 &= \frac{(\xi^1-\xi_1^1)(\xi_1^2-\xi_2^2) + (\xi_2^1-\xi_1^1)(\xi^2-\xi_1^2)}{\Delta} \end{aligned}\right\} \tag{7.2.3}$$

where

$$\left.\begin{aligned}\Delta &= \xi_1^1(\xi_2^2-\xi_3^2) + \xi_2^1(\xi_3^2-\xi_1^2) + \xi_3^1(\xi_1^2-\xi_2^2)\\ &= (\xi_1^1-\xi_3^1)(\xi_2^2-\xi_3^2) + (\xi_2^1-\xi_3^1)(\xi_3^2-\xi_1^2)\\ &= (\overrightarrow{\Sigma_1\Sigma_3} \times \overrightarrow{\Sigma_2\Sigma_3}) \bullet (\vec{e}_1 \times \vec{e}_2) \neq 0 .\end{aligned}\right\} \tag{7.2.4}$$

7.3 - Eccentricity parameters

In order to define conveniently the orthogonal direction to the side $\Sigma_{i-1}\,\Sigma_{i+1}$, we introduce the eccentricity parameters $\eta_i \in \mathbb{R}$, $i=1,2,3$, i.e.

$$\eta_i = 2\,\frac{\overline{c_i b_i}}{\Sigma_{i-1}\Sigma_{i+1}} = \frac{(\ell_{i-1})^2-(\ell_{i+1})^2}{\ell_i^2}\quad , \quad i=1,2,3 . \tag{7.3.1}$$

By b_i, c_i, ℓ_i, we respectively denote the midside of $\Sigma_{i-1}\Sigma_{i+1}$, the orthogonal projection of the point Σ_i on the side $\Sigma_{i+1}\Sigma_{i+2}$ and the length of the side $\Sigma_{i-1}\Sigma_{i+1}$.

7.4 - Local and global degrees of freedom

In the definition of the degrees of freedom of the reduced HCT-triangle we have only used the geometry of the triangle. These degrees of freedom are called *local*. To assemble the contributions of each element, it is convenient to replace the direction of the sides of the triangle by the fixed directions given by the vectors $\vec{e}_1$, $\vec{e}_2$, and next, to normalize.

Thus, to the three local degrees of freedom attached to the vertex Σ_i, i.e.,

$$v \to \{v(\Sigma_i), Dv(\Sigma_i)(\Sigma_{i-1}-\Sigma_i), Dv(\Sigma_i)(\Sigma_{i+1}-\Sigma_i)\} \tag{7.4.1}$$

we associate the three global degrees of freedom

$$v \to \{v(\Sigma_i)\ ,\ Dv(\Sigma_i)\vec{e}_1 = \frac{\partial v}{\partial \xi^1}(\Sigma_i)\ ,\ Dv(\Sigma_i)\vec{e}_2 = \frac{\partial v}{\partial \xi^2}(\Sigma_i)\}. \tag{7.4.2}$$

We obtain the relations

$$\begin{bmatrix} Dv(\Sigma_i)(\Sigma_{i-1}-\Sigma_i) \\ Dv(\Sigma_i)(\Sigma_{i+1}-\Sigma_i) \end{bmatrix} = \begin{bmatrix} \xi^1_{i-1}-\xi^1_i & \xi^2_{i-1}-\xi^2_i \\ \xi^1_{i+1}-\xi^1_i & \xi^2_{i+1}-\xi^2_i \end{bmatrix} \begin{bmatrix} \frac{\partial v}{\partial \xi^1}(\Sigma_i) \\ \frac{\partial v}{\partial \xi^2}(\Sigma_i) \end{bmatrix} \tag{7.4.3}$$

Let us set

$$\left.\begin{aligned} [DLGL_i(v)] = [&v(\Sigma_i)v(\Sigma_{i+1})v(\Sigma_{i+2})\frac{\partial v}{\partial \xi^1}(\Sigma_i)\frac{\partial v}{\partial \xi^2}(\Sigma_i)\frac{\partial v}{\partial \xi^1}(\Sigma_{i+1})\frac{\partial v}{\partial \xi^2}(\Sigma_{i+1}) \\ &\frac{\partial v}{\partial \xi^1}(\Sigma_{i+2})\frac{\partial v}{\partial \xi^2}(\Sigma_{i+2})] \end{aligned}\right\} \tag{7.4.4}$$

$$\left.\begin{aligned} [DLLC_i(v)] = [&v(\Sigma_i)v(\Sigma_{i+1})v(\Sigma_{i+2})Dv(\Sigma_i)(\Sigma_{i+2}-\Sigma_i)Dv(\Sigma_i)(\Sigma_{i+1}-\Sigma_i) \\ &Dv(\Sigma_{i+1})(\Sigma_i-\Sigma_{i+1})Dv(\Sigma_{i+1})(\Sigma_{i+2}-\Sigma_{i+1}) \\ &Dv(\Sigma_{i+2})(\Sigma_{i+1}-\Sigma_{i+2})Dv(\Sigma_{i+2})(\Sigma_i-\Sigma_{i+2})] \end{aligned}\right\} \tag{7.4.5}$$

$$[d_i]_{2\times 2} = \begin{bmatrix} \xi^1_{i-1}-\xi^1_i & \xi^1_{i+1}-\xi^1_i \\ \xi^2_{i-1}-\xi^2_i & \xi^2_{i+1}-\xi^2_i \end{bmatrix} \quad , \quad i=1,2,3\ . \tag{7.4.6}$$

Then, we obtain

$$[DLLC_i(v)]_{(1,9)} = [DLGL_i(v)]_{1,9}\ [D_i]_{(9,9)} \tag{7.4.7}$$

with

$$[D_i] = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & 0 \\ & & & d_i & & \\ & 0 & & & d_{i+1} & \\ & & & & & d_{i+2} \end{bmatrix} \tag{7.4.8}$$

7.5 - Basis functions

In BERNADOU-HASSAN [20], we have shown that the basis functions (shape functions) for the *general* reduced HCT-triangle are given by the figure 7.5.1 and are associated to the following interpolating function $\pi_{K_i} v$ of a (sufficiently smooth) function v on the subtriangle K_i, $i=1,2,3$.

$$\left.\begin{aligned}\pi_{K_i} v = {} & v(\Sigma_i)r^0_{i,i} + v(\Sigma_{i+1})r^0_{i,i+1} + v(\Sigma_{i+2})r^0_{i,i+2} \\ & + Dv(\Sigma_i)(\Sigma_{i+2}-\Sigma_i)r^1_{i,i,i+2} + Dv(\Sigma_i)(\Sigma_{i+1}-\Sigma_i)r^1_{i,i,i+1} \\ & + Dv(\Sigma_{i+1})(\Sigma_i-\Sigma_{i+1})r^1_{i,i+1,i} + Dv(\Sigma_{i+1})(\Sigma_{i+2}-\Sigma_{i+1})r^1_{i,i+1,i+2} \\ & + Dv(\Sigma_{i+2})(\Sigma_{i+1}-\Sigma_{i+2})r^1_{i,i+2,i+1} + Dv(\Sigma_{i+2})(\Sigma_i-\Sigma_{i+2})r^1_{i,i+2,i}\end{aligned}\right\rbrace \quad (7.5.1$$

Figure 7.5.1 can be condensed as follows :

$$[R_i]_{9\times 1} = [A_i]_{9\times 10} \quad [\Lambda_i]_{10\times 1} . \tag{7.5.2}$$

7.6 - Interpolation of a function v and partial derivatives of the interpolating function

Let $\pi_{K_i} v$ be the interpolating function of a function v on the subtriangle K_i. The relations (7.4.5) (7.5.1) reveal that

$$\pi_{K_i} v = [DLLC_i(v)]_{1\times 9} \; [R_i]_{9\times 1} , \tag{7.6.1}$$

or equivalently, using relations (7.4.7) and (7.5.2)

$$\pi_{K_i} v = [DLGL_i(v)]_{1\times 9} \; [D_i]_{9\times 9} \; [A_i]_{9\times 10} \; [\Lambda_i]_{10\times 1} . \tag{7.6.2}$$

$$
\begin{bmatrix}
r^{o}_{i,i} \\
r^{o}_{i,i+1} \\
r^{o}_{i,i+2} \\
r^{1}_{i,i,i+2} \\
r^{1}_{i,i,i+1} \\
r^{1}_{i,i+1,i} \\
r^{1}_{i,i+1,i+2} \\
r^{1}_{i,i+2,i+1} \\
r^{1}_{i,i+2,i}
\end{bmatrix}
=
\begin{bmatrix}
-\frac{1}{2}(\eta_{i+1}-\eta_{i+2}) & 0 & 0 & \frac{3}{2}(3+\eta_{i+1}) & \frac{3}{2}(3-\eta_{i+2}) & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}(1-2\eta_i-\eta_{i+2}) & 1 & 0 & -\frac{3}{2}(1-\eta_i) & \frac{3}{2}(\eta_i+\eta_{i+2}) & 3 & 3 & 0 & 0 & 3(1-\eta_i) \\
\frac{1}{2}(1+2\eta_i+\eta_{i+1}) & 0 & 1 & -\frac{3}{2}(\eta_i+\eta_{i+1}) & -\frac{3}{2}(1+\eta_i) & 0 & 0 & 3 & 3 & 3(1+\eta_i) \\
-\frac{1}{4}(1+\eta_{i+1}) & 0 & 0 & \frac{1}{4}(5+3\eta_{i+1}) & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4}(1-\eta_{i+2}) & 0 & 0 & \frac{1}{2} & \frac{1}{4}(5-3\eta_{i+2}) & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4}(1-\eta_{i+2}) & 0 & 0 & -\frac{1}{2} & -\frac{1}{4}(1-3\eta_{i+2}) & 1 & 0 & 0 & 0 & 1 \\
-\frac{1}{2}\eta_i & 0 & 0 & -\frac{1}{4}(1-3\eta_i) & \frac{1}{4}(1+3\eta_i) & 0 & 1 & 0 & 0 & \frac{1}{2}(1-3\eta_i) \\
\frac{1}{2}\eta_i & 0 & 0 & \frac{1}{4}(1-3\eta_i) & -\frac{1}{4}(1+3\eta_i) & 0 & 0 & 1 & 0 & \frac{1}{2}(1+3\eta_i) \\
\frac{1}{4}(1+\eta_{i+1}) & 0 & 0 & -\frac{1}{4}(1+3\eta_{i+1}) & -\frac{1}{2} & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\lambda_i^3 \\
\lambda_{i+1}^3 \\
\lambda_{i+2}^3 \\
\lambda_i^2\lambda_{i+2} \\
\lambda_i^2\lambda_{i+1} \\
\lambda_{i+1}^2\lambda_i \\
\lambda_{i+1}^2\lambda_{i+2} \\
\lambda_{i+2}^2\lambda_{i+1} \\
\lambda_{i+2}^2\lambda_i \\
\lambda_i\lambda_{i+1}\lambda_{i+2}
\end{bmatrix}
$$

Figure 7.5.1 : Basis functions associated to subtriangle K_i for the reduced-HCT finite element

Then, the derivatives of the interpolating function $\pi_{K_i} v$ are given by the expressions

$$\left.\begin{aligned} \partial_\alpha \pi_{K_i} v &= [DLGL_i(v)]\,[D_i]\,[A_i]\,[\partial_\alpha \Lambda_i]\ , \quad [\partial_\alpha \Lambda_i] = [\frac{\partial \Lambda_i}{\partial \xi^\alpha}]\ , \\ \partial_{\alpha\beta} \pi_{K_i} v &= [DLGL_i(v)]\,[D_i]\,[A_i]\,[\partial_{\alpha\beta} \Lambda_i]\ , \quad [\partial_{\alpha\beta} \Lambda_i] = [\frac{\partial^2 \Lambda_i}{\partial \xi^\alpha \partial \xi^\beta}]\ . \end{aligned}\right\} \quad (7.6.3)$$

The column matrices $[\partial_1 \Lambda_i]$ and $[\partial_2 \Lambda_i]$ are defined from the matrices $[\frac{\partial \Lambda_i}{\partial \lambda_j}]$, j=1,2,3 (cf. tableau 7.6.1) as follows

$$\left.\begin{aligned} [\partial_1 \Lambda_i] &= \frac{1}{\Delta}\,\{(\xi_2^2-\xi_3^2)[\frac{\partial \Lambda_i}{\partial \lambda_1}] + (\xi_3^2-\xi_1^2)[\frac{\partial \Lambda_i}{\partial \lambda_2}] + (\xi_1^2-\xi_2^2)[\frac{\partial \Lambda_i}{\partial \lambda_3}]\}, \\ [\partial_2 \Lambda_i] &= \frac{1}{\Delta}\,\{(\xi_3^1-\xi_2^1)[\frac{\partial \Lambda_i}{\partial \lambda_1}] + (\xi_1^1-\xi_3^1)[\frac{\partial \Lambda_i}{\partial \lambda_2}] + (\xi_2^1-\xi_1^1)[\frac{\partial \Lambda_i}{\partial \lambda_3}]\}. \end{aligned}\right\} \quad (7.6.4)$$

By analogy, we obtain $[\partial_{\alpha\beta} \Lambda_i] = [\frac{\partial^2 \Lambda_i}{\partial \xi^\alpha \partial \xi^\beta}]$ as :

$$\begin{aligned} [\partial_{11} \Lambda_i] = \frac{1}{\Delta^2}\,\{&(\xi_2^2-\xi_3^2)^2[\frac{\partial^2 \Lambda_i}{(\partial \lambda_1)^2}] + (\xi_3^2-\xi_1^2)^2[\frac{\partial^2 \Lambda_i}{(\partial \lambda_2)^2}] \\ &+ (\xi_1^2-\xi_2^2)^2\,[\frac{\partial^2 \Lambda_i}{(\partial \lambda_3)^2}] + 2(\xi_2^2-\xi_3^2)(\xi_3^2-\xi_1^2)\,[\frac{\partial^2 \Lambda_i}{\partial \lambda_1 \partial \lambda_2}] \\ &+ 2(\xi_3^2-\xi_1^2)(\xi_1^2-\xi_2^2)[\frac{\partial^2 \Lambda_i}{\partial \lambda_2 \partial \lambda_3}] + 2(\xi_1^2-\xi_2^2)(\xi_2^2-\xi_3^2)\,[\frac{\partial^2 \Lambda_i}{\partial \lambda_3 \partial \lambda_1}]\} \end{aligned} \quad (7.6.5)$$

$$\left.\begin{aligned}
[\partial_{12}\Lambda_i] = \frac{1}{\Delta^2}\{&(\xi_2^2-\xi_3^2)(\xi_3^1-\xi_2^1)[\frac{\partial^2\Lambda_i}{(\partial\lambda_1)^2}] + (\xi_3^2-\xi_1^2)(\xi_1^1-\xi_3^1)\,[\frac{\partial^2\Lambda_i}{(\partial\lambda_2)^2}] \\
&+ (\xi_1^2-\xi_2^2)(\xi_2^1-\xi_1^1)\,[\frac{\partial^2\Lambda_i}{(\partial\lambda_3)^2}] \\
&+ ((\xi_2^2-\xi_3^2)(\xi_1^1-\xi_3^1) + (\xi_3^2-\xi_1^2)(\xi_3^1-\xi_2^1))\,[\frac{\partial^2\Lambda_i}{\partial\lambda_1\partial\lambda_2}] \\
&+ ((\xi_3^2-\xi_1^2)(\xi_2^1-\xi_1^1) + (\xi_1^2-\xi_2^2)(\xi_1^1-\xi_3^1))\,[\frac{\partial^2\Lambda_i}{\partial\lambda_2\partial\lambda_3}] \\
&+ ((\xi_1^2-\xi_2^2)(\xi_3^1-\xi_2^1) + (\xi_2^2-\xi_3^2)(\xi_2^1-\xi_1^1))\,[\frac{\partial^2\Lambda_i}{\partial\lambda_3\partial\lambda_1}]\} \\
[\partial_{22}\Lambda_i] = \frac{1}{\Delta^2}\{&(\xi_3^1-\xi_2^1)^2\,[\frac{\partial^2\Lambda_i}{(\partial\lambda_1)^2}] + (\xi_1^1-\xi_3^1)^2\,[\frac{\partial^2\Lambda_i}{(\partial\lambda_2)^2}] \\
&+ (\xi_2^1-\xi_1^1)^2\,[\frac{\partial^2\Lambda_i}{(\partial\lambda_3)^2}] + 2(\xi_3^1-\xi_2^1)(\xi_1^1-\xi_3^1)\,[\frac{\partial^2\Lambda_i}{\partial\lambda_1\partial\lambda_2}] \\
&+ 2(\xi_1^1-\xi_3^1)(\xi_2^1-\xi_1^1)\,[\frac{\partial^2\Lambda_i}{\partial\lambda_2\partial\lambda_3}] + 2(\xi_2^1-\xi_1^1)(\xi_3^1-\xi_2^1)\,[\frac{\partial^2\Lambda_i}{\partial\lambda_3\partial\lambda_1}]\}
\end{aligned}\right\}\quad(7.6.5)$$

7.7 - Implementation of the triangle of type (1) (i.e., P_1-Lagrange)

Let $\bar{\omega}_K\, v$ be the interpolating function of a function v on the triangle K, using the triangle of type (1) associated to P_1-Lagrange interpolation. Similarly to (7.6.2) (7.6.3), we obtain

$$\bar{\omega}_K\, v = [\mathrm{d1g1}(v)]_{1\times3}\,[\lambda]_{3\times1} \qquad (7.7.1)$$

where

$$[\mathrm{dlgl}(v)]_{1\times 3} = [v(\Sigma_1) \quad v(\Sigma_2) \quad v(\Sigma_3)] \tag{7.7.2}$$

$$^t[\lambda]_{1\times 3} = [\lambda_1 \quad \lambda_2 \quad \lambda_3] \tag{7.7.3}$$

Similarly to (7.6.4), we obtain

$$^t[\partial_1\lambda] = \frac{1}{\Delta}[\xi_2^2-\xi_3^2 \quad \xi_3^2-\xi_1^2 \quad \xi_1^2-\xi_2^2] \tag{7.7.4}$$

$$^t[\partial_2\lambda] = \frac{1}{\Delta}[\xi_3^1-\xi_2^1 \quad \xi_1^1-\xi_3^1 \quad \xi_2^1-\xi_1^1] \tag{7.7.5}$$

8 - IMPLEMENTATION OF THE FACET STIFFNESS AND SECOND MEMBER MATRICES

8.1 - Implementation of the facet stiffness matrix

For clarity, we begin to describe the implementation of the facet stiffness matrix directly derived from the expression (4.3.3) :

$$\left.\begin{aligned}\tilde{a}_{hK}(\vec{\tilde{u}}_h,\vec{\tilde{v}}_h) = \int_K \frac{Ee}{1-\nu^2}\{(1-\nu)\tilde{\gamma}^{\alpha}_{h\beta}(\vec{\tilde{u}}_h)\tilde{\gamma}^{\beta}_{h\alpha}(\vec{\tilde{v}}_h)+\nu\tilde{\gamma}^{\alpha}_{h\alpha}(\vec{\tilde{u}}_h)\tilde{\gamma}^{\beta}_{h\beta}(\vec{\tilde{v}}_h)\\ +\frac{e^2}{12}[(1-\nu)\tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{u}}_h)\tilde{\rho}^{\beta}_{h\alpha}(\vec{\tilde{v}}_h)+\nu\tilde{\rho}^{\alpha}_{h\alpha}(\vec{\tilde{u}}_h)\tilde{\rho}^{\beta}_{h\beta}(\vec{\tilde{v}}_h)]\}\sqrt{a_h}\, d\xi^1 d\xi^2.\end{aligned}\right\}\tag{8.1.1}$$

For the time being we leave aside the problem of compatibility relations and the only work on the facet K. Since the functions $\vec{\tilde{u}}_h$ and $\vec{\tilde{v}}_h$ belong to the space $\vec{\tilde{X}}_h$ - see (4.2.1) (4.2.3) (4.2.4) - the relations (7.6.2) (7.7.1) reveal that on the triangle $K = \bigcup_{i=1}^{3} K_i$ we have

$$\tilde{u}_{h\alpha} = [d1g1(\tilde{u}_{h\alpha})]_{1\times 3}\,[\lambda]_{3\times 1} \tag{8.1.2}$$

$$\tilde{u}_{h3}\big|_{K_i} = [DLGL_i(\tilde{u}_{h3})]_{1\times 9}\,[D_i]_{9\times 9}\,[A_i]_{9\times 10}\,[\Lambda_i]_{10\times 1}\,. \tag{8.1.3}$$

Moreover from (8.1.2) (7.7.4) (7.7.5) on the one hand and from (8.1.3) (7.6.3) on the other hand, we derive

$$\tilde{u}_{h\alpha,\beta} = [d1g1(\tilde{u}_{h\alpha})]_{1\times 3}\,[\partial_\beta\lambda] \tag{8.1.4}$$

$$\tilde{u}_{h3,\alpha}\big|_{K_i} = [DLGL_i(\tilde{u}_{h3})]_{1\times 9}\,[D_i]_{9\times 9}\,[A_i]_{9\times 10}\,[\partial_\alpha\Lambda_i]_{10\times 1} \tag{8.1.5}$$

$$\tilde{u}_{h3,\alpha\beta}\big|_{K_i} = [DLGL_i(\tilde{u}_{h3})]_{1\times 9}\,[D_i]_{9\times 9}\,[A_i]_{9\times 10}\,[\partial_{\alpha\beta}\Lambda_i]_{10\times 1} \tag{8.1.6}$$

Expression of $\tilde{\gamma}^{\alpha}_{h\beta}(\tilde{u}_h)$

From (4.1.4) (4.2.11) we obtain

$$\tilde{\gamma}^{\alpha}_{h\beta}(\vec{\tilde{u}}_h) = a_h^{\alpha\lambda}\tilde{\gamma}_{h\beta\lambda} = \frac{1}{2}\, a_h^{\alpha\lambda}(\tilde{u}_{h\beta,\lambda} + \tilde{u}_{h\lambda,\beta}) \tag{8.1.7}$$

or

$$\tilde{\gamma}^{\alpha}_{h\beta}(\vec{\tilde{u}}_h) = M^{\alpha}_{h\beta}\, \tilde{U}_{h12} \tag{8.1.8}$$

where

$$M^{\alpha}_{h\beta} = [0;a_h^{1\alpha}\delta_{\beta 1}\ ;\ \frac{1}{2}\, a_h^{\alpha\lambda}\Delta_{\lambda\beta}\ ;\ 0\ ;\ \frac{1}{2}\, a_h^{\alpha\lambda}\Delta_{\lambda\beta}\ ;\ a_h^{2\alpha}\delta_{\beta 2}] \tag{8.1.9}$$

$$\Delta_{\lambda\beta} = 0 \quad \text{if} \quad \lambda = \beta \quad , \quad \Delta_{\lambda\beta} = 1 \quad \text{if } \lambda \neq \beta, \tag{8.1.10}$$

$${}^t\tilde{U}_{h12} = [\tilde{u}_{h1} \quad \tilde{u}_{h1,1} \quad \tilde{u}_{h1,2} \quad \tilde{u}_{h2} \quad \tilde{u}_{h2,1} \quad \tilde{u}_{h2,2}] \ . \tag{8.1.11}$$

Let us set

$$[DG12(\vec{\tilde{u}}_h)]_{1\times 6} = [d1g1(\tilde{u}_{h1});d1g1(\tilde{u}_{h2})]_{1\times 6} \ . \tag{8.1.12}$$

Then, using relations (8.1.2) and (8.1.4) we can write

$${}^t\tilde{U}_{h12} = [DG12(\vec{\tilde{u}}_h)]_{1\times 6}\ [LAMBD12]_{6\times 6} \tag{8.1.13}$$

with

$$[LAMBD12]_{6\times 6} = \left[\begin{array}{ccc|ccc} \lambda & \partial_1\lambda & \partial_2\lambda & & \text{o} & \\ \hline & \text{o} & & \lambda & \partial_1\lambda & \partial_2\lambda \end{array}\right] \ . \tag{8.1.14}$$

Hence the relation (8.1.8) can be rewritten as

$$\tilde{\gamma}^{\alpha}_{h\beta}(\vec{\tilde{u}}_h) = M^{\alpha}_{h\beta}\ {}^t[LAMBD12]\ {}^t[DG12(\vec{\tilde{u}}_h)] \ . \tag{8.1.15}$$

<u>Expression of</u> $\tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{u}}_h)$:

From (4.1.4) (4.2.12) we obtain

$$\tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{u}}_h) = a_h^{\alpha\lambda}\tilde{\rho}_{h\beta\lambda} = a_h^{\alpha\lambda}\tilde{v}_{h3,\lambda\beta} \tag{8.1.16}$$

or, equivalently

$$\tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{u}}_h) = B^{\alpha}_{h\beta}\ \tilde{U}_{h3} \tag{8.1.17}$$

where

$$B^{\alpha}_{h\beta} = [0 \quad 0 \quad 0 \quad a_h^{1\alpha}\delta_{\beta 1} \quad a_h^{\alpha\lambda}\Delta_{\lambda\beta} \quad a_h^{2\alpha}\delta_{\beta 2}] \tag{8.1.18}$$

$${}^t\tilde{U}_{h3} = [\tilde{u}_{h3} \quad \tilde{u}_{h3,1} \quad \tilde{u}_{h3,2} \quad \tilde{u}_{h3,11} \quad \tilde{u}_{h3,12} \quad \tilde{u}_{h3,22}] \ . \tag{8.1.19}$$

Let us set

$$[DA_i]_{9\times 10} = [D_i]_{9\times 9}\ [A_i]_{9\times 10} \tag{8.1.20}$$

$$[LAMBD3_i]_{10\times 6} = [\Lambda_i \quad \partial_1\Lambda_i \quad \partial_2\Lambda_i \quad \partial_{11}\Lambda_i \quad \partial_{12}\Lambda_i \quad \partial_{22}\Lambda_i] \ . \tag{8.1.21}$$

Then, the relation (8.1.19) becomes

$${}^t\tilde{U}_{h3}\big|_{K_i} = [DLGL_i(\tilde{u}_{h3})]_{1\times 9}\ [DA_i]_{9\times 10}\ [LAMBD3_i]_{10\times 6} \tag{8.1.22}$$

so that

$$\tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{u}}_h)\big|_{K_i} = B^{\alpha}_{h\beta}\ {}^t[LAMBD3_i]\ {}^t[DA_i]\ {}^t[DLGL_i(\tilde{u}_{h3})] \tag{8.1.23}$$

Expression of the facet stiffness matrix

Combining the relation (8.1.1) with the expressions (8.1.15) and (8.1.23), we obtain

$$\tilde{a}_{hK}(\vec{\tilde{u}}_h,\vec{\tilde{v}}_h) = [DG12(\vec{\tilde{u}}_h)](\int_K [LAMBD12][M_h]\ {}^t[LAMBD12]d\xi^1 d\xi^2)\,{}^t[DG12(\vec{\tilde{v}}_h)]$$

$$+ \sum_{i=1}^{3} ([DLGL_i(\tilde{u}_{h3})][DA_i]\{\int_{K_i} [LAMBD3_i][B_h]\,{}^t[LAMBD3_i]d\xi^1 d\xi^2\}\ {}^t[DA_i]\ {}^t[DLGL_i(\tilde{v}_{h3})]) \qquad (8.1.24)$$

where

$$[M_h]_{6\times 6} = \frac{Ee\sqrt{a_h}}{1-\nu^2}\{(1-\nu)\ {}^tM^{\alpha}_{h\beta}\ M^{\beta}_{h\alpha} + \nu\ {}^tM^{\alpha}_{h\alpha}\ M^{\beta}_{h\beta}\} \qquad (8.1.25)$$

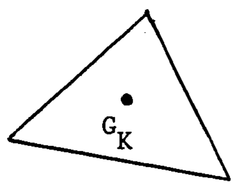
$$[B_h]_{6\times 6} = \frac{Ee^3\sqrt{a_h}}{12(1-\nu^2)}\{(1-\nu)\ {}^tB^{\alpha}_{h\beta}\ B^{\beta}_{h\alpha} + \nu\,{}^tB^{\alpha}_{h\alpha}\ B^{\beta}_{h\beta}\}\ . \qquad (8.1.26)$$

At the beginning of this section we have mentionned that we have considered the element stiffness matrix associated to the expression (4.3.3). In fact, we consider practically the expression (4.3.14) taking into account the effect of numerical integration. With obvious notations, the relations (4.3.14) and (8.1.24) reveal that

$$\left.\begin{aligned}
&a^*_{hK}(\vec{\tilde{u}}_h,\vec{\tilde{v}}_h) = \\
&[DG12(\vec{\tilde{u}}_h)](\sum_{\ell_1=1}^{L_1} \omega_{\ell_1,K}([LAMBD12][M_h]\ {}^t[LAMBD12])(b_{\ell_1,K})\ {}^t[DG12(\vec{\tilde{v}}_h)] \\
&+ \sum_{i=1}^{3} ([DLGL_i(\tilde{u}_{h3})][DA_i]\{\sum_{\ell_2=1}^{L_2} \omega_{\ell_2,K_i}([LAMBD3_i][B_h]\ {}^t[LAMBD3_i]) \\
&\qquad (b_{\ell_2,K_i})\}\,{}^t[DA_i]\ {}^t[DLGL_i(\tilde{v}_{h3})]).
\end{aligned}\right\} \qquad (8.1.27)$$

According to (6.1.1) and (6.1.2), we can use following numerical integration schemes in order to compute respectively first and second terms of the above relation.

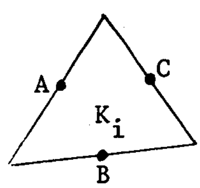
<u>SCHEME 8.1.1</u> : integration of extension terms; exact for P_0.



$$\int_K f(x)dx \sim f(G_K)\mathrm{meas}(K) \tag{8.1.28}$$

<u>SCHEME 8.1.2</u> : integration of flexion terms; exact for P_2.

This scheme has to be used on every subtriangle K_i of triangle K.



$$\int_{K_i} f(x)dx \sim \frac{1}{3}\{f(A)+f(B)+f(C)\}\mathrm{meas}(K_i) \tag{8.1.29}$$

8.2 - <u>Implementation of the facet second member matrix</u>

Similarly to section 8.1, we introduce the following expressions $\tilde{f}_{hK}$ and $f^\star_{hK}$ respectively associated to the relations (4.3.4) and (4.3.15), i.e.,

$$\tilde{f}_{hK}(\vec{\tilde{v}}_h) = \int_K \vec{p}\,\vec{\tilde{v}}_h \sqrt{a_h}\, d\xi^1 d\xi^2 \ , \tag{8.2.1}$$

$$\left.\begin{aligned} f^\star_{hK}(\vec{\tilde{v}}_h) = & \sum_{\ell_1=1}^{L_1} \omega_{\ell_1,K} \left(\{\vec{p}.\vec{a}_h^{\alpha}\}\ \tilde{v}_{h\alpha}\sqrt{a_h}\right)(b_{\ell_1,K}) \\ & + \sum_{i=1}^{3} \sum_{\ell_2=1}^{L_2} \omega_{\ell_2,K_i} \left(\{\vec{p}.\vec{a}_h^{\,3}\}\ \tilde{v}_{h3}\sqrt{a_h}\right)(b_{\ell_2,K_i}) \ . \end{aligned}\right\} \tag{8.2.2}$$

Setting

$$ {}^{t}\tilde{P}_{h12} = \sqrt{a_h}\ [\tilde{p}_h^1 \quad 0 \quad 0 \quad \tilde{p}_h^2 \quad 0 \quad 0] \tag{8.2.3}$$

$$ {}^{t}\tilde{P}_{h3} = \sqrt{a_h}\ [\tilde{p}_h^3 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] \tag{8.2.4}$$

with

$$ \tilde{p}_h^i = \vec{p} \cdot \vec{a}_h^i , \tag{8.2.5}$$

we derive from (8.1.11) (8.1.19)

$$ \tilde{f}_{hK}(\vec{\tilde{v}}_h) = \int_K ({}^{t}\tilde{P}_{h12}\ \tilde{V}_{h12} + {}^{t}\tilde{P}_{h3}\ \tilde{V}_{h3})d\xi^1 d\xi^2 . \tag{8.2.6}$$

Then, the relations (8.1.13) (8.1.22) give

$$\left.\begin{aligned} \tilde{f}_{hK}(\vec{\tilde{v}}_h) &= \{ \int_K {}^{t}\tilde{P}_{h12}\ {}^{t}[LAMBD12]d\xi^1 d\xi^2 \}\ {}^{t}[DG12(\vec{\tilde{v}}_h)] \\ &+ \sum_{i=1}^{3} \{ (\int_{K_i} {}^{t}\tilde{P}_{h3}\ {}^{t}[LAMBD3_i]d\xi^1 d\xi^2)\ {}^{t}[DA_i]{}^{t}[DLGL_i(\tilde{v}_{h3})] \} . \end{aligned}\right\} \tag{8.2.7}$$

Finally, we immediately obtain

$$\begin{aligned} f^{*}_{hK}(\vec{\tilde{v}}_h) &= \{ \sum_{\ell_1=1}^{L_1} \omega_{\ell_1,K}\ ({}^{t}\tilde{P}_{h12}\ {}^{t}[LAMBD12])(b_{\ell_1,K}) \}\ {}^{t}[DG12(\vec{\tilde{v}}_h)] \\ &+ \sum_{i=1}^{3} \{ (\sum_{\ell_2=1}^{L_2} \omega_{\ell_2,K_i}\ ({}^{t}\tilde{P}_{h3}\ {}^{t}[LAMBD3_i](b_{\ell_2,K_i})))\ {}^{t}[DA_i]\ {}^{t}[DLGL_i(\tilde{v}_{h3})] \} \end{aligned} \tag{8.2.8}$$

As numerical integration schemes, we can use (8.1.28) and (8.1.29) in order to compute respectively the first and the second terms of second member of the above relation.

8.3 - Implementation of the compatibility relations

The relations (8.1.27) and (8.2.8) permit to compute the stiffness and second member matrices *facet by facet*. To derive the corresponding *global* stiffness and second member matrices, we have to assemble. A convenient way consists in

(i) using the compatibility relations to obtain the expressions of $\vec{\tilde{u}}_h$ (resp. $\vec{\tilde{v}}_h$) as functions of $\vec{u}_h$ (resp. $\vec{v}_h$),

(ii) assembling the facet contributions so that the global matrices are referred to the unknowns $\vec{u}_h$.

In this section we consider the point (i). The second point (ii) will be the object of next paragraph.

Let us set for any triangle $K \in \mathcal{C}_h$ and for any $\vec{u}_h \in \vec{X}_h$

$$\left.\begin{aligned}[DG(\vec{u}_h)] = [&u_{h1}(\Sigma_1) \quad u_{h1}(\Sigma_2) \quad u_{h1}(\Sigma_3) \quad u_{h2}(\Sigma_1) \quad u_{h2}(\Sigma_2) \quad u_{h2}(\Sigma_3) \\ &u_{h3}(\Sigma_1) \quad u_{h3}(\Sigma_2) \quad u_{h3}(\Sigma_3) \quad u_{h3,1}(\Sigma_1) \quad u_{h3,2}(\Sigma_1) \\ &u_{h3,1}(\Sigma_2) \quad u_{h3,2}(\Sigma_2) \quad u_{h3,1}(\Sigma_3) \quad u_{h3,2}(\Sigma_3)] \,.\end{aligned}\right\} \tag{8.3.1}$$

Similarly, for any $\vec{\tilde{u}}_h \in \vec{\tilde{X}}_h$, we set

$$[DG(\vec{\tilde{u}}_h)] = [DG12(\vec{\tilde{u}}_h) \; ; \; DLGL(\tilde{u}_{h3})] \tag{8.3.2}$$

where the matrices $[DG12(\vec{\tilde{u}}_h)]$ and $[DLGL(\vec{\tilde{u}}_{h3})] \equiv [DLGL_1(\tilde{u}_{h3})]$ are defined by (8.1.12) and (7.4.4), respectively. Then, using compatibility relations we want to determine the matrix $[TILD]_{15\times 15}$ so that, for any functions $\vec{\tilde{u}}_h \in \vec{\tilde{X}}_h$ and $\vec{u}_h \in \vec{X}_h$ in correspondence through the bijection F_h defined by theorem 4.2.1, we have :

$$[DG(\vec{\tilde{u}}_h)]_{1\times 15} = [DG(\vec{u}_h)]_{1\times 15} \; [TILD]_{15\times 15} \;. \tag{8.3.3}$$

From (4.2.21) (4.2.23), we obtain

$$\vec{u}_h(\Sigma_j) = \vec{u}_h(\Sigma_j) \quad , \quad j=1,2,3 .$$

Observing that on the triangle K, we have

$$\vec{u}_h(\Sigma_j) = \tilde{u}_{hi}(\Sigma_j)\vec{a}_h^i \tag{8.3.4}$$

$$\vec{u}_h(\Sigma_j) = u_{hk}(\Sigma_j)\vec{a}^k(\Sigma_j) \quad , \quad j=1,2,3, \tag{8.3.5}$$

we get

$$\tilde{u}_{hi}(\Sigma_j) = d_{hi}^k(\Sigma_j)u_{hk}(\Sigma_j) \quad , \quad j=1,2,3, \tag{8.3.6}$$

where, for any point $M \in K$

$$d_{hi}^k(M) = \vec{a}^k(M) \cdot \vec{a}_{hi} . \tag{8.3.7}$$

Let us record that, due to the definition of the mapping $\vec{\phi}_h\big|_K \in (P_1(K))^3$, the basis vectors $\vec{a}_h^i$ and $\vec{a}_{hi}$ are constants on all the triangle K.

Now, let us determine $\tilde{u}_{h3,\alpha}(\Sigma_i)$ as functions of the elements of $[DG(\vec{u}_h)]$. Firstly we observe that $\tilde{u}_{h\alpha} \in \tilde{X}_{h1}$ and the relations (7.7.1) (7.7.4) (7.7.5) imply that the functions $\tilde{u}_{h\alpha,\beta}$ are constant on all the triangle K and equal to

$$\tilde{u}_{h\alpha,\beta} = [d1g1(\tilde{u}_{h\alpha})]\,[\partial_\beta\lambda] . \tag{8.3.8}$$

Hence the relation (4.2.24) give the expression of the function $\tilde{\omega}_h^3$ which is also a constant triangle by triangle, i.e., on the triangle K :

$$\tilde{\omega}_h^3 = \frac{1}{2\sqrt{a_h}}\left([d1g1(\tilde{u}_{h2})]\,[\partial_1\lambda] - [d1g1(\tilde{u}_{h1})]\,[\partial_2\lambda]\right) . \tag{8.3.9}$$

Secondly, the compatibility relations (4.2.25) involve

$$\tilde{\omega}_h^\beta(\Sigma_j)\vec{a}_{h\beta} \cdot \vec{a}^\lambda(\Sigma_j) = B^\lambda(\Sigma_j) - \tilde{\omega}_h^3\,\vec{a}_{h3} \cdot \vec{a}^\lambda(\Sigma_j) \quad , \quad j=1,2,3 \ ,$$

or equivalently, by using (8.3.7)

$$d_{h\beta}^\lambda(\Sigma_j)\;\tilde{\omega}_h^\beta(\Sigma_j) = B^\lambda(\Sigma_j) - d_{h3}^\lambda(\Sigma_j)\tilde{\omega}_h^3 \quad , \quad j=1,2,3 \ . \tag{8.3.10}$$

Let us denote

$$c^\lambda(\Sigma_j) = B^\lambda(\Sigma_j) - d_{h3}^\lambda(\Sigma_j)\;\tilde{\omega}_h^3 \tag{8.3.11}$$

$$d_h(\Sigma_j) = d_{h1}^1(\Sigma_j)\;d_{h2}^2(\Sigma_j) - d_{h1}^2(\Sigma_j)\;d_{h2}^1(\Sigma_j) \ . \tag{8.3.12}$$

From section 4.2 we have $d_h(\Sigma_j) \neq 0$. Then, the relations (8.3.10) to (8.3.12) imply

$$\tilde{\omega}_h^\lambda(\Sigma_j) = \frac{1}{d_h(\Sigma_j)}\;e^{\lambda\nu}\;e_{\alpha\beta}\;c^\alpha(\Sigma_j)\;d_{h\nu}^\beta(\Sigma_j) \quad , \quad j=1,2,3 \ .$$

Finally, from (4.2.26), we derive (compare with relation just before (5.2.45))

$$\tilde{u}_{h3,\alpha}(\Sigma_j) = \frac{\sqrt{a_h}}{d_h(\Sigma_j)}\;e_{\lambda\beta}\;c^\lambda(\Sigma_j)\;d_{h\alpha}^\beta(\Sigma_j) \quad , \quad j=1,2,3 \ . \tag{8.3.13}$$

From (8.3.6) (8.3.9), we have

$$\tilde{\omega}_h^3 = \frac{1}{2\sqrt{a_h}}\,[DG(\vec{u}_h)]_{1\times 15}\;[T^3]_{15\times 1} \tag{8.3.14}$$

with

$$[T^3]_{15\times 1} = \begin{bmatrix} D_{h1} & D_{h2} \\ 0 & 0 \end{bmatrix}_{15\times 6} \begin{bmatrix} -\partial_2\lambda \\ \partial_1\lambda \end{bmatrix}_{6\times 1} \tag{8.3.15}$$

and

$$[D_{hi}]_{9\times 3} = \begin{bmatrix} d^1_{hi}(\Sigma_1) & 0 & 0 \\ 0 & d^1_{hi}(\Sigma_2) & 0 \\ 0 & & d^1_{hi}(\Sigma_3) \\ d^2_{hi}(\Sigma_1) & 0 & 0 \\ 0 & d^2_{hi}(\Sigma_2) & 0 \\ 0 & 0 & d^2_{hi}(\Sigma_3) \\ d^3_{hi}(\Sigma_1) & 0 & 0 \\ 0 & d^3_{hi}(\Sigma_2) & 0 \\ 0 & 0 & d^3_{hi}(\Sigma_3) \end{bmatrix} \tag{8.3.16}$$

From (4.2.22) we have

$$B^\lambda(\Sigma_j) = [DG(\vec{u}_h)]_{1\times 15}\ [T^\lambda_j]_{15\times 1} \tag{8.3.17}$$

with

$$\left.\begin{array}{l} {}^t[T^\lambda_1]_{1\times 15} = e^{\lambda\beta}\,[\ \dfrac{b^1_\beta}{\sqrt{a}}(\Sigma_1)\quad 0\quad 0\quad \dfrac{b^2_\beta}{\sqrt{a}}(\Sigma_1)\quad 0\quad 0\quad 0\quad 0\quad 0 \\ \qquad\qquad \dfrac{\delta_{1\beta}}{\sqrt{a(\Sigma_1)}}\quad \dfrac{\delta_{2\beta}}{\sqrt{a(\Sigma_1)}}\quad 0\quad 0\quad 0\quad 0\ ] \end{array}\right\} \tag{8.3.18}$$

$$\left.\begin{array}{l} {}^t[T^\lambda_2]_{1\times 15} = e^{\lambda\beta}\,[0\quad \dfrac{b^1_\beta}{\sqrt{a}}(\Sigma_2)\quad 0\quad 0\quad \dfrac{b^2_\beta}{\sqrt{a}}(\Sigma_2)\quad 0\quad 0\quad 0\quad 0 \\ \qquad\qquad 0\quad 0\quad \dfrac{\delta_{1\beta}}{\sqrt{a(\Sigma_2)}}\quad \dfrac{\delta_{2\beta}}{\sqrt{a(\Sigma_2)}}\quad 0\quad 0\ ] \end{array}\right\} \tag{8.3.19}$$

$$\left.\begin{aligned} {}^t[T_3^\lambda]_{1\times15} = e^{\lambda\beta}\,[0 \quad 0 \quad \frac{b_\beta^1}{\sqrt{a}}(\Sigma_3) \quad 0 \quad 0 \quad \frac{b_\beta^2}{\sqrt{a}}(\Sigma_3) \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \quad \frac{\delta_{1\beta}}{\sqrt{a(\Sigma_3)}} \quad \frac{\delta_{2\beta}}{\sqrt{a(\Sigma_3)}}\,]\;. \end{aligned}\right\} \tag{8.3.20}$$

Then, combining (8.3.6) (8.3.11) (8.3.13) (8.3.14) (8.3.16) we obtain

$$[DG(\vec{\tilde{u}}_h)]_{1\times15} = [DG(\vec{u}_h)]_{1\times15}\,[TILD]_{15\times15} \tag{8.3.21}$$

with

$$[TILD]_{15\times15} = \left[\begin{array}{c|c|c|c|c|c|c|c|c} Dh_1 & D_{h2} & D_{h3} & & & & & & \\ & & & R_{11} & R_{21} & R_{12} & R_{22} & R_{13} & R_{23} \\ 0 & 0 & 0 & & & & & & \end{array}\right] \tag{8.3.22}$$

and

$$[R_{\alpha j}]_{15\times1} = \frac{d_{h\alpha}^\beta(\Sigma_j)}{d_h(\Sigma_j)}\,e_{\lambda\beta}\,\{\sqrt{a_h}\,[T_j^\lambda]_{15\times1} - \frac{1}{2}\,d_{h3}^\lambda(\Sigma_j)\,[T^3]_{15\times1}\}\;. \tag{8.3.23}$$

9 - DERIVATION OF THE GLOBAL STIFFNESS AND SECOND MEMBER MATRICES

9.1 - Derivation of the global stiffness matrix

Using the compatibility relations obtained triangle by triangle through the relations (8.3.21), we substitute the *global* unknowns $[DG(\vec{u}_h)]$ to the *facet* unknowns $[DG(\tilde{\vec{u}}_h)]$.

For convenience, we set

$$[J_{12}]_{15\times 6} = \begin{bmatrix} I_6 \\ \hline 0_{9\times 6} \end{bmatrix} , \tag{9.1.1}$$

$$[J_3]_{15\times 9} = \begin{bmatrix} 0_{6\times 9} \\ \hline I_9 \end{bmatrix} . \tag{9.1.2}$$

Then the relations (7.4.4) (8.1.12) (8.3.1) (8.3.2) (8.3.21) reveal that

$$\begin{aligned} &[DG12(\tilde{\vec{u}}_h)]_{1\times 6} = [DG(\vec{u}_h)]_{1\times 15}\, [TILD]_{15\times 15}\, [J_{12}]_{15\times 6} \\ &[DLGL(\tilde{u}_{h3})]_{1\times 9} = [DG(\vec{u}_h)]_{1\times 15}\, [TILD]_{15\times 15}\, [J_3]_{15\times 9} \quad . \end{aligned} \tag{9.1.3}$$

Observe from (7.4.4) and (8.3.2) that

$$[DLGL_i(\tilde{v}_h)] = [DLGL(\tilde{v}_h)][K_i]_{9\times 9} \tag{9.1.4}$$

with

$$[K_i] = \begin{bmatrix} K_{i11} & 0 \\ 0 & K_{i22} \end{bmatrix} , \quad [K_{i11}]_{3\times 3} = \begin{bmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{2i} & \delta_{3i} & \delta_{1i} \\ \delta_{3i} & \delta_{1i} & \delta_{2i} \end{bmatrix} .$$

and

$$[K_{i22}]_{6\times 6} = \begin{bmatrix} \delta_{1i} & 0 & \delta_{2i} & 0 & \delta_{3i} & 0 \\ 0 & \delta_{1i} & 0 & \delta_{2i} & 0 & \delta_{3i} \\ \delta_{2i} & 0 & \delta_{3i} & 0 & \delta_{1i} & 0 \\ 0 & \delta_{2i} & 0 & \delta_{3i} & 0 & \delta_{1i} \\ \delta_{3i} & 0 & \delta_{1i} & 0 & \delta_{2i} & 0 \\ 0 & \delta_{3i} & 0 & \delta_{1i} & 0 & \delta_{2i} \end{bmatrix}$$

so that

$$[DLGL_i(\tilde{u}_{h3})]_{1\times 9} = [DG(\vec{u}_h)]_{1\times 15}\,[TILD]_{15\times 15}\,[K_{3i}]_{15\times 9} \tag{9.1.5}$$

with

$$[K_{3i}]_{15\times 9} = \begin{bmatrix} 0_{6\times 9} \\ \hline K_i \end{bmatrix}. \tag{9.1.6}$$

From (4.3.16) and (8.1.27), we finally obtain

$$\left.\begin{aligned} b_h(\vec{u}_h,\vec{v}_h) = \sum_{k\in\mathcal{T}_h} \Big[& [DG(\vec{u}_h)]_{1\times 15}\,[TILD]_{15\times 15} \\ & \{\, [J_{12}]_{15\times 6} \Big(\sum_{\ell_1=1}^{L_1} \omega_{\ell_1,K}\,([LAMBD12]_{6\times 6}\,[M_h]_{6\times 6}\,{}^t[LAMBD12]) \\ & \qquad (b_{\ell_1,K})\Big)\ {}^t[J_{12}] \\ & + \sum_{i=1}^{3} ([K_{3i}]_{15\times 9}\,[DA_i]_{9\times 10}\,\{ \sum_{\ell_2=1}^{L_2} \omega_{\ell_2,K_i}\,([LAMBD3_i]_{10\times 6} \\ & [B_h]_{6\times 6}\,{}^t[LAMBD3_i])\,(b_{\ell_2,K_i})\}\ {}^t[DA_i]\ {}^t[K_{3i}])\ \} \\ & {}^t[TILD]\ {}^t[DG(\vec{v}_h)]\Big] \end{aligned}\right\} \tag{9.1.7}$$

Thus, assembling the contributions of every triangle $K \in \mathcal{T}_h$ we get the *global stiffness matrix*.

9.2 - Derivation of the global second member

Similarly to section 9.1, the relations (4.3.17) and (8.2.8) reveal that

$$\left.\begin{aligned} g_h(\vec{v}_h) = \sum_{K \in \mathcal{T}_h} \Big[(\{ \sum_{\ell_1=1}^{L_1} & \omega_{\ell_1,K} \, ({}^t\tilde{P}_{h12} \; {}^t[\mathrm{LAMBD12}])(b_{\ell_1,K}) \} \; {}^t[J_{12}] \\ & + \sum_{i=1}^{3} \{\{ \sum_{\ell_2=1}^{L_2} \omega_{\ell_2,K_i} \, ({}^t\tilde{P}_{h3} \; {}^t[\mathrm{LAMBD3}])(b_{\ell_2,K_i})\} {}^t[DA_i] \; {}^t[K_{3i}]\}) \\ & \qquad {}^t[\mathrm{TILD}] \; {}^t[DG(\vec{v}_h)] \Big] . \end{aligned}\right\} \quad (9.2.1)$$

By assembling the contributions of every triangle $K \in \mathcal{T}_h$ we get the *global second member matrix*:

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